On Magnetohydrodynamic Stability in the Low Pressure Limit *, **

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The stability of arbitrary toroidal magnetohydrostatic equilibria is investigated near the limit of a vacuum magnetic field. Nonsingular expansions in powers of $\beta \sim p/B^2$ are considered to all orders. Order by order minimization of the energy variaiton shows that *any* orders are positive semidefinite in the nullspace of all lower orders if either the vacuum field has shear or (for an identically vanishing rotational number) the interchange stability condition is satisfied. The question of sufficiency for stability at low β of these conditions is discussed. While sufficiency of shear can be refuted by considering localized modes, this question remains open for the interchange stability condition.

1. Introduction

A large body of all previous work on ideal MHD stability theory is based on expansions in powers of parameters describing the equilibrium. Such expansions have hitherto been carried out only up to the lowest nontrivial order. Conditions were thus derived which are *necessary* for stability if the expansion parameter is sufficiently small. They were, moreover, often believed to be also sufficient. An objection against sufficiency is that more stringent necessary conditions could possibly arise in some higher orders.

The purpose of the present paper is twofold. Expanding the energy variation in the usual (nonsingular) way in powers of the plasma β, and minimizing it order by order, we firstly show that the full information obtainable by this procedure already appears in the lowest nontrivial order, and that the above objection therefore does not apply. In other words, we derive conditions which are sufficient for *any* orders of the energy variation to be nonnegative on the nullspace of all lower orders. Secondly, we discuss the question of sufficiency for *stability* at low β of such conditions. It turns out that certain modes which become singular in the limit $\beta \rightarrow 0$ can be unstable at arbitrarily low β even if such conditions are satisfied. General arguments indicate, moreover, that conditions arising from *any*, e. g. singular, β-expansions can fail to be sufficient.

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2. General Preparations

We consider arbitrary toroidal magnetohydrostatic equilibria, subject to

$$\nabla p = \boldsymbol{j} \times \boldsymbol{B}, \quad \boldsymbol{j} = \operatorname{curl} \boldsymbol{B}, \quad \operatorname{div} \boldsymbol{B} = 0 \qquad (2.1)$$

where the symbols have their usual meaning. Assuming that the fluid is in contact with a perfectly conducting rigid wall, we can write the energy variation [1, 2] as

$$W = \tfrac{1}{2} \int \mathrm{d}\tau \; \boldsymbol{\xi} \cdot \mathcal{F} \boldsymbol{\xi}, \qquad (2.2)$$

where

$$\mathcal{F} \boldsymbol{\xi} = \boldsymbol{B} \times \operatorname{curl} \boldsymbol{Q} - \boldsymbol{j} \times \boldsymbol{Q} - \nabla (\boldsymbol{\xi} \cdot \nabla p + \gamma \, p \operatorname{div} \boldsymbol{\xi}), \qquad (2.3)$$

$$\boldsymbol{Q} = \operatorname{curl}(\boldsymbol{\xi} \times \boldsymbol{B}). \qquad (2.4)$$

Exponentially growing modes are present, if, and only if, $W < 0$ for some displacements $\boldsymbol{\xi}$ which are tangential at the wall.

When expanding the functional W in powers of a parameter in a straightforward manner, we would encounter great difficulties in high orders. These are caused by the presence of the lower orders of the neutral diplacements, which can, of course, not be excluded by minimization. We can conveniently get rid of the neutral displacements from the beginning by introducing certain constraints, which are, in turn, most conveniently formulated in terms of a set (v, ϑ, ζ) of co-ordinates with the following properties [3, 4]:

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1. v is the volume enclosed by the toroidal surfaces of constant pressure,
2. ϑ and ζ are angle-like quantities increasing by unity once the short and the long way round,
3. the Jacobian is unity,
4. the contra-variant components of the vectors $\boldsymbol{B}$ and $\boldsymbol{j}$ are surface quantities (i. e. functions of v).

Such co-ordinates exist as long as $\mathrm{d}p/\mathrm{d}v \neq 0$. Assuming this we restrict our considerations to doubly connected configurations with simply nested pressure surfaces, in which the pressure gradient is nowhere zero except on the magnetic axis, where it drops to zero linearly with the distance.

We will impose constraints on the contra-variant components

$$X = \boldsymbol{\xi} \cdot \nabla v\,, \quad Y = \boldsymbol{\xi} \cdot \nabla \vartheta\,, \quad Z = \boldsymbol{\xi} \cdot \nabla \zeta \tag{2.5}$$

of the displacement vector.
These constraints depend on the behaviour of the rotational number. If the latter is nonconstant, then the trivially neutral displacements have the form

$$\boldsymbol{\xi}_0 = Y_0(v)\ \boldsymbol{e}_\vartheta + Z_0(v)\ \boldsymbol{e}_\zeta\,, \tag{2.6}$$

where $\boldsymbol{e}_\vartheta$ and $\boldsymbol{e}_\zeta$ are co-variant basis vectors. Since the operator $\mathcal{F}$, Eq. (2.3), is linear and symmetric, and since $\mathcal{F}\,\boldsymbol{\xi}_0 = 0$, we have

$$W[\boldsymbol{\xi} + \xi_0] = W[\boldsymbol{\xi}]\,. \tag{2.7}$$

Hence we do not alter the infimum of W if we replace any given displacement $\boldsymbol{\xi}$ by another displacement $\boldsymbol{\xi} + \boldsymbol{\xi}_0$, where Y_0 and Z_0 in Eq. (2.6) are chosen such that

$$\int \mathrm{d}\vartheta\, \mathrm{d}\zeta\, Y = 0, \qquad \int \mathrm{d}\vartheta\, \mathrm{d}\zeta\, Z = 0\,. \tag{2.8}$$

If the rotational number is identically zero, then the trivially neutral displacements have the more general form

$$\xi_0 = Y_0(v)\ \boldsymbol{e}_\vartheta + Z_0(v, \vartheta)\ \boldsymbol{e}_\zeta\,, \tag{2.9}$$

and the adequate (more restrictive) constraints are

$$\int \mathrm{d}\vartheta\, \mathrm{d}\zeta\, Y = 0, \qquad \int \mathrm{d}\zeta\, Z = 0. \tag{2.10}$$

The cases of a constant nonvanishing rotational number will not be considered.

In terms of the co-ordinates (v, ϑ, ζ), the energy variation is

$$W = \tfrac{1}{2} \int_0^V \mathrm{d}v \int_0^1 \mathrm{d}\vartheta \int_0^1 \mathrm{d}\zeta\, \{g_{ik}\, Q^i\, Q^k + \dot{p}\, X\, \mathrm{div}\, \boldsymbol{\xi} + \varepsilon_{klm}\, j^k\, \xi^l\, Q^m + \gamma\, p\, (\mathrm{div}\, \boldsymbol{\xi})^2\}\,, \tag{2.11}$$

where V is the total volume of the plasma, $g_{ik} = \boldsymbol{e}_i \cdot \boldsymbol{e}_k$, ε_{klm} is the fully anti-symmetric unit tensor,

$$Q^v = \dot{\chi}\, \frac{\partial X}{\partial \vartheta} + \dot{\psi}\, \frac{\partial X}{\partial \zeta}\,, \tag{2.12}$$

$$Q^\vartheta = \dot{\psi}\, \frac{\partial Y}{\partial \zeta} - \dot{\chi}\, \frac{\partial Z}{\partial \zeta} - \frac{\partial (\dot{\chi} X)}{\partial v}\,, \tag{2.13}$$

$$Q^\zeta = -\dot{\psi}\, \frac{\partial Y}{\partial \vartheta} + \dot{\chi}\, \frac{\partial Z}{\partial \vartheta}\, \frac{\partial (\dot{\psi} X)}{\partial v}\,, \tag{2.14}$$

$$\mathrm{div}\, \boldsymbol{\xi} = \frac{\partial X}{\partial v} + \frac{\partial Y}{\partial \vartheta} + \frac{\partial Z}{\partial \zeta}\,, \tag{2.15}$$

and summations are to be performed over twice occuring indices. $\psi(v)$ and $\chi(v)$ are the magnetic fluxes the long and the short way round contained within the pressure surface of volume v, and dots denote derivatives with respect to v.

3. The β-expansion

We consider a family of equilibrium configurations depending on a parameter β such that in the limit $\beta \to 0$ the magnetic field is a vacuum field produced by external currents, and $p = O(\beta)$. Hence $j^k = O(\beta)$. If we assume that the boundary conditions (i. e. the position and the shape of the wall) are independent of β, then the diagonal elements of the metric dyadic g_{ik} are all of the same order, namely $O(1)$. Assuming further that in the limit $\beta \to 0$ the equilibrium is nonsingular, we have $\dot{p} = O(\beta)$. The components of $\boldsymbol{\xi}$ will depend on β even if $\boldsymbol{\xi}$ itself does not. We thus write

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \sum_{\nu \geq 0} \beta^\nu \begin{pmatrix} X^{(\nu)}(v, \vartheta, \zeta) \\ Y^{(\nu)}(v, \vartheta, \zeta) \\ Z^{(\nu)}(v, \vartheta, \zeta) \end{pmatrix}, \tag{3.1}$$

and use the analoguous notation for equilibrium quantities. The energy variation can now be expanded into a series of the form

$$W = \sum_{\nu \geq 0} \beta^\nu\, W^{(\nu)}\ [X^{(0)}, Y^{(0)}, Z^{(0)}, \dots, X^{(\nu)}, Y^{(\nu)}, Z^{(\nu)}]\,. \tag{3.2}$$

The constraints (2.8) and (2.9) may be imposed separately in each order.

We wish to minimize the series (3.2) to all orders. Its lowest order,

$$W^{(0)} = \tfrac{1}{2} \int \mathrm{d}v\, \mathrm{d}\vartheta\, \mathrm{d}\zeta\, g_{ik}^{(0)}\, Q^{i(0)}\, Q^{k(0)}, \tag{3.3}$$

is nonnegative. It is zero if, and only if, $Q^i = O(\beta)$. In view of the Eqs. (2.12) – (2.14), this reads

$$\dot{\chi}^{(0)} \frac{\partial X^{(0)}}{\partial \vartheta} + \dot{\psi}^{(0)} \frac{\partial X^{(0)}}{\partial \zeta} = 0, \tag{3.4}$$

$$\dot{\psi}^{(0)} \frac{\partial Y^{(0)}}{\partial \zeta} - \dot{\chi}^{(0)} \frac{\partial Z^{(0)}}{\partial \zeta} - \frac{\partial(\dot{\chi}^{(0)} X^{(0)})}{\partial v} = 0, \tag{3.5}$$

$$\dot{\psi}^{(0)} \frac{\partial Y^{(0)}}{\partial \vartheta} - \dot{\chi}^{(0)} \frac{\partial Z^{(0)}}{\partial \vartheta} + \frac{\partial(\dot{\psi}^{(0)} X^{(0)})}{\partial v} = 0. \tag{3.6}$$

The implications of these relations are different in the two cases to be considered.

A. Configuration with Shear

We assume that the quantity

$$S = \dot{\chi}\ddot{\psi} - \dot{\psi}\ddot{\chi}, \tag{3.7}$$

which is sometimes called "shear", is $O(1)$, i. e. that $S^{(0)} \neq 0$ except on isolated surfaces. In this case the only continuous and single-valued (i. e. periodic) solutions of the Eqs. (3.4) – (3.6) satisfying the constraints (2.8) are

$$\left.\begin{aligned} X^{(0)} &= 0, \\ Y^{(0)} &= \dot{\chi}^{(0)} F, \\ Z^{(0)} &= \dot{\psi}^{(0)} F, \end{aligned}\right\} \tag{3.8}$$

where F is an arbitrary single-valued continuous function with $\int \mathrm{d}\vartheta\, \mathrm{d}\zeta\, F = 0$. Hence, in the nullspace of $W^{(0)}$,

$$W^{(1)} = \tfrac{1}{2} \int \mathrm{d}v\, \mathrm{d}\vartheta\, \mathrm{d}\zeta\, \gamma\, p^{(1)} \left(\dot{\chi}^{(0)} \frac{\partial F}{\partial \vartheta} + \dot{\psi}^{(0)} \frac{\partial F}{\partial \zeta} \right)^2, \tag{3.9}$$

which again is nonnegative. It is zero if, and only if, $F = 0$. Hence in the nullspace of $W^{(0)}$ and $W^{(1)}$,

$$X^{(0)} \equiv 0, \quad Y^{(0)} \equiv 0, \quad Z^{(0)} \equiv 0. \tag{3.10}$$

and the functional form of $W^{(2)}$ is precisely the same as that of $W^{(0)}$ was before, the trial functions $X^{(0)}$, $Y^{(0)}$, and $Z^{(0)}$ merely being replaced by the functions $X^{(1)}$, $Y^{(1)}$, and $Z^{(1)}$. Repeating the above procedure one thus encounters the same situation after every two steps. Hence the condition $S^{(0)} \neq 0$ is sufficient for *any* order of W to be positive semi-definite on the nullspace of all lower orders.

B. Zero Rotational Number

Here $\dot{\chi} \equiv 0$, and Eqs. (3.4) – (3.6), together with the lowest order of Eq. (2.15), imply

$$\partial X^{(0)}/\partial\zeta = 0, \quad \partial Y^{(0)}/\partial\zeta = 0,$$
$$\operatorname{div} \boldsymbol{\xi} = -\frac{\ddot{\psi}^{(0)}}{\dot{\psi}^{(0)}} X^{(0)} + \frac{\partial Z^{(0)}}{\partial \zeta} + O(\beta). \tag{3.11}$$

The first order of the energy variation is then

$$W^{(1)} = \tfrac{1}{2} \int \mathrm{d}v\, \mathrm{d}\vartheta\, \mathrm{d}\zeta \left\{ \left[-\dot{p}^{(1)} \frac{\ddot{\psi}^{(0)}}{\dot{\psi}^{(0)}} + \gamma\, p^{(1)} \left(\frac{\ddot{\psi}^{(0)}}{\dot{\psi}^{(0)}} \right)^2 \right] (X^{(0)})^2 + \gamma\, p^{(1)} \left(\frac{\partial Z^{(0)}}{\partial \zeta} \right)^2 \right\}, \tag{3.12}$$

which is nonnegative if, and only if,

$$-\dot{p}^{(1)} \frac{\ddot{\psi}^{(0)}}{\dot{\psi}^{(0)}} + \gamma\, p^{(1)} \left(\frac{\ddot{\psi}^{(0)}}{\dot{\psi}^{(0)}} \right)^2 \geqq 0 \tag{3.13}$$

on every pressure surface. To proceed to higher orders we assume that the condition (3.13) is satisfied. For simplicity we omit the case in which it is only marginally satisfied throughout finite regions. Minimization of $W^{(1)}$ then yields

$$X^{(0)} \equiv 0, \quad \partial Z^{(0)}/\partial\zeta \equiv 0,$$

and the last of the Eqs. (3.11) implies $\partial Y^{(0)}/\partial\vartheta \equiv 0$. We can now, this time with the help of the constraints (2.10), again conclude that

$$X^{(0)} \equiv 0, \quad Y^{(0)} \equiv 0, \quad Z^{(0)} \equiv 0 \tag{3.14}$$

in the nullspace of $W^{(0)}$ and $W^{(1)}$. The same argument as that employed in the case $S^{(0)} \neq 0$ then shows that the condition (3.13) is sufficient for the positiveness of *all* orders of the energy variation.

This condition is the lowest order of the well-known necessary and sufficient interchange stability condition[5–7]. Its relevance to the *lowest* nontrivial order in the β-expansion was previously known[2, 8].

4. Discussion

In summary we state: The lowest nonvanishing order of the energy variation is positive, if either of the following two conditions is satisfied.

1. The vacuum magnetic field has shear.

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2. The total field has zero rotational number, and the vacuum field satisfies, together with the lowest order pressure, the interchange stability condition (3.13).

One might therefore conclude that each one of the above two conditions is sufficient for stability if β is below some *positive* critical value. The following argument, which in essence is due to GRAD [9], indicates, however, that this conclusion need not hold: One can merely conclude that for every *fixed* displacement vector $\boldsymbol{\xi}$, a positive value $\beta[\boldsymbol{\xi}]$ of the parameter β exists such that $W[\boldsymbol{\xi}] \geqq 0$ if $\beta \leqq \beta[\boldsymbol{\xi}]$. It may then happen that $\beta[\boldsymbol{\xi}]$ approaches zero when $\boldsymbol{\xi}$ is varied over its whole possible range. In this case the critical β is zero, and no conclusion is possible. So far the question is nevertheless still open because no examples to the contrary have yet been given.

We now give a class of examples to the contrary for the case $S = O(1)$. Here the well-known *necessary* condition for stability to localized modes [4, 10, 11] is automatically satisfied at low β everywhere except on the magnetic axis, where it reduces to $\dot{p}^{(1)} \ddot{\psi}^{(0)} \leqq 0$ (MERCIER [12]). This condition can obviously be violated. Instabilities can thus occur at arbitrarily low β, no matter whether $S^{(0)} \neq 0$ or not. A rough estimate shows that these modes are localized within a range of width $O(\beta)$ at a distance $O(\beta)$ from the axis, and have a growth rate $O(\beta)$. Such modes have, of course, been excluded by our assumption that the displacements are nonsingular in the limit $\beta \rightarrow 0$. Their instability could therefore possibly be recovered within the framework of some *singular* expansion. From the above general argument it is clear, on the other hand, that one can never be sure to get the correct answer by *any* expansion. It should also be mentioned that this failure has nothing to do with any singular behaviour of the equilibrium. This can be seen by considering helically symmetric equilibria because these can have any desired properties.

As to the case of zero rotational number, the criterion for stability to localized modes [13] does not provide an example to the contrary, because it converges *uniformly* to the interchange stability condition (3.13) in the low β limit. It follows, on the other hand, from exact sufficient stability criteria (see notes [14, 15]) that the slightly more restrictive condition $\dot{p}^{(1)} \ddot{\psi}^{(0)} < 0$ is sufficient for the existence of a positive critical β, provided that the current is zero on the magnetic axis. Strictly speaking, the question whether or not the interchange stability condition can, at low β, fail to be sufficient for stability nevertheless still remains open.

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